

# Relation between the usual and the entanglement temperature, in a simple quantum system

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## Abstract

We develop a thermodynamical theory to describe the behavior of the entanglement between a single two-level atom with a single mode of the electromagnetic field. The resonant Jaynes-Cummings model is used to study both the entanglement thermodynamics, in particular the entanglement temperature, and its connection with the average number of photons in the optical cavity. We find that this entanglement temperature has a strong dependence with the initial conditions of the atom. We show that the entanglement temperature between the photons and the atom defined in this work is the same temperature obtained within the Jaynes-Cummings model at finite temperature developed in the Thermo-Field Dynamics formalism.

*Key words:* Quantum computation; Quantum information

PACS: 03.65.Yz, 03.67.-a

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## 1 Introduction

Concepts such as thermodynamic equilibrium seem impossible to reconcile with the idea of isolated quantum systems since such systems follow unitary evolutions and do not reach a final stationary equilibrium state. Of course, a completely isolated quantum system is an idealization, constructed as a help to understand some phenomena displayed by real systems which may be regarded as approximately isolated. However, we recently [1,2,3,4] introduced the concept of temperature for an isolated quantum system which evolves in a composite Hilbert space. To do this we consider the quantum walk on the line (QW) (see [5] and references therein). The QW is a natural generalization of the classical random walk in the frame of quantum computation and quantum information processing and it receiving much attention recently [6,7,8,9].

In our above mentioned works, we have developed a thermodynamic theory to describe the behavior of the entanglement between the coin and position degrees of freedom of the QW. Henceforth, we call “entanglement temperature” to the temperature associated to the entanglement between different degrees of freedom of an isolated quantum system. We have shown that, in spite of the evolution being unitary, in the QW a steady state is established after a Markovian transient stage. Those studies suggest that, if a quantum dynamics develops in a composite Hilbert space (*i.e.* the tensor product of several sub-spaces), then the behavior of an operator that belongs only to one of the sub-spaces may camouflage the unitary character of the global evolution. However it is not clear what is the relation between the usual temperature and the entanglement temperature. This question cannot be answered using the QW because it is an abstract mathematical model; to do this we need a real physical model where both temperatures, the usual and the entanglement temperature, emerge naturally. In order to answer it, we have here chosen one of the simplest and most interesting quantum models, the one known as the Jaynes-Cummings model (JCM) [10,11], that studies the interaction between radiation and matter.

The JCM considers the interaction between a single two-level atom with a single mode of the electromagnetic field. The coupling between the atom and the field is characterized by a Rabi frequency, and a loss of excitation in the atom appears as a gain in excitation of the field. The collapse and the eventual revival of the Rabi oscillation, described by the analytical solution of the JCM, is a direct evidence of the quantum nature of radiation. The use of the JCM has permitted to elucidate basic properties of quantum entanglement as well as some aspects of the relationship between classical and quantum physics. Since it was proposed, the phenomenon has been of permanent interest in the quantum theory of interactions. About 30 years ago it was found that the model exhibits highly non-classic behavior, and the possibility of experimental realizations appeared. The relative simplicity of the JCM and its extensions has drawn much attention in the physics community and, more recently, in the field of the quantum computing [15,16].

Also in the 80’s, the Thermo Field Dynamics (TFD) formalism [12,13] was applied to the JCM. The TFD is a method, developed in the 70’s by Takahashi and Umezawa [14], for describing Quantum Mechanical systems at finite temperature. Using this method, it is possible to describe the statistical average of an observable at finite temperature as a pure state expectation value. Thus, within the TFD formalism, one does not need to deal with a mixed state, which is a statistical ensemble of pure states at finite temperature. In return for the above advantage, the TFD introduces the so-called tilde particles corresponding to ordinary particles, thus doubling the dimension of the Hilbert space associated to the system. In the TFD method the ordinary particles and the introduced tilde particles represent the dynamical degrees of freedom and

the thermal degrees of freedom, respectively.

In the present work we connect, within the JCM, the TFD thermodynamics with the entanglement thermodynamics presented in our previous works [1,2,3]. The paper is organized as follows. In the next section we review the usual JCM and study the photon thermodynamics in that model. In third section we develop the entanglement thermodynamics for the JCM and study its connection to the TFD-defined temperature. Finally, in the last section we draw some conclusions.

## 2 Jaynes-Cummings model

We consider the ordinary JCM [11], composed by a single two-state atom in an optical cavity, interacting with a single quantized mode, with frequency  $\omega$ . The Hilbert space of the JCM has the form of a tensor product

$$\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_A, \quad (1)$$

where the photon space,  $\mathcal{H}_N$ , is spanned by the unitary orthonormal vectors of the photon number state  $\{|n\rangle\}$ , and the atom space,  $\mathcal{H}_A$ , is spanned by the two orthonormal quantum states  $\{|e\rangle, |f\rangle\}$  that represent the excited and fundamental states of the atom, respectively. Note that the set  $\{|n, e\rangle, |n, f\rangle\}$ , where  $|n, e\rangle = |n\rangle|e\rangle$  and  $|n, f\rangle = |n\rangle|f\rangle$ , is an orthonormal base in the JCM Hilbert space.

In this model, if the atom excitation frequency  $\omega_a$  is close to  $\omega$ , then the system is near the resonance and it is possible to use the rotating wave approximation. In this case, and removing the field vacuum energy, the system Hamiltonian is

$$H = \hbar\omega a^\dagger a + \frac{\hbar}{2}\omega_a\sigma_z + \frac{\hbar}{2}g(a^\dagger\sigma_- + a\sigma_+), \quad (2)$$

where  $a^\dagger$  and  $a$  are the photon creation and annihilation operators respectively, and act on the photon number state  $|n\rangle$ . The radiation-matter coupling constant  $g$  is fixed by physical considerations such as the cavity volume and the atomic dipole moment. The raising and lowering operators are defined by

$$\begin{aligned} \sigma_+ &\equiv |e\rangle\langle f|, \\ \sigma_- &\equiv |f\rangle\langle e|, \end{aligned} \quad (3)$$

and the  $z$  Pauli operator by

$$\sigma_z \equiv |e\rangle\langle e| - |f\rangle\langle f| = [\sigma_+, \sigma_-], \quad (4)$$

and act on the atom states. Then the Hamiltonian, Eq. (2), is such that each photon creation is accompanied by an atomic de-excitation, and each photon annihilation by an atomic excitation. For a given photon number value  $n$ , Eq. (2) has the eigenvalues

$$E_{\pm}(n) = \hbar\omega(n + \frac{1}{2}) \pm \frac{1}{2}\hbar\Omega_n(\delta), \quad (5)$$

where, for a specific detuning parameter  $\delta \equiv \omega - \omega_a$ ,

$$\Omega_n(\delta) = \sqrt{\delta^2 + (n+1)g^2}, \quad (6)$$

is the Rabi frequency. The corresponding eigenvectors, called “dressed states”, are given by

$$\begin{aligned} |n+\rangle &= \cos\theta_n |n, e\rangle + \sin\theta_n |n+1, f\rangle, \\ |n-\rangle &= \cos\theta_n |n+1, f\rangle - \sin\theta_n |n, e\rangle, \end{aligned} \quad (7)$$

where

$$\tan(2\theta_n) = \frac{g\sqrt{n+1}}{\delta}. \quad (8)$$

Let us call  $|\Psi(t)\rangle$  the wave function of the JCM system. Its dynamics is given by the time-dependent Schrödinger equation and we will procure general solutions of the form

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} \left( a_n^+ e^{-\frac{iE_+(n)t}{\hbar}} |n+\rangle + a_n^- e^{-\frac{iE_-(n)t}{\hbar}} |n-\rangle \right), \quad (9)$$

where the coefficients  $a_n^+$  and  $a_n^-$  are fixed by the initial conditions. In this work we consider the case of a separable atom-photon initial state. We assume that the initial state of the atom is arbitrary and that the initial photon number follows some initial coherent distribution. However, we must be careful to eliminate the  $|0\rangle|f\rangle$  state in the initial condition because this state can not be built from the dressed states, see Eq. (7). More specifically, we take initial conditions of the form

$$|\Psi(0)\rangle = \mathcal{N} \sum_{n=0}^{\infty} C_n |n\rangle \left( |e\rangle \cos \frac{\gamma}{2} + |f\rangle e^{i\varphi} \sin \frac{\gamma}{2} \right) - \mathcal{N} C_0 |0\rangle |f\rangle e^{i\varphi} \sin \frac{\gamma}{2}, \quad (10)$$

where the two parameters  $\gamma \in [0, \pi]$  and  $\varphi \in [0, 2\pi]$  define the initial state of the atom characterized by a point on the generalized Bloch’s sphere and  $\mathcal{N}$  is a normalization constant,

$$\mathcal{N} = \frac{1}{\sqrt{1 - |C_0|^2 \sin^2 \frac{\gamma}{2}}}. \quad (11)$$

Here we take  $\{|C_n|^2\}$  as the initial coherent photon distribution inside of the cavity without atoms. Note that  $\sum_{n=0}^{\infty} |C_n|^2 = 1$ , then  $\bar{n} \equiv \sum_{n=0}^{\infty} n |C_n|^2$  is

the initial average number of photons. Using Eqs.(9),(10) and projecting on the base  $\{|n, e\rangle, |n, f\rangle\}$ , it is straightforward to obtain the connection between  $\{a_n^+, a_n^-\}$  and the initial state of the atom and photon number,

$$\begin{aligned} a_n^+ &= \mathcal{N} \left( C_n \cos \theta_n \cos \frac{\gamma}{2} + C_{n+1} \sin \theta_n e^{i\varphi} \sin \frac{\gamma}{2} \right), \\ a_n^- &= \mathcal{N} \left( C_{n+1} \cos \theta_n e^{i\varphi} \sin \frac{\gamma}{2} - C_n \sin \theta_n \cos \frac{\gamma}{2} \right). \end{aligned} \quad (12)$$

For this system, as is the case with most closed quantum systems, the probability distribution does not converge in time. However we can use a natural notion of convergence in the quantum case, if we define the “limiting distribution” as the asymptotic limit of the average of the probability distributions at time

$$P \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(t) dt. \quad (13)$$

This definition captures the amount of time the system spends in each average state, and moreover, it corresponds to the natural concept of sampling from the system, since if one measures the system state at a random time chosen from the interval  $[0, t]$ , the resulting distribution is exactly the average distribution. Applying the limit of the average to the previous definitions, we calculate the probability to obtain  $n$  photons independently of the state of the atom, *i.e.*  $P(n) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\langle n | \Psi(t) \rangle|^2 dt$ . After a straightforward calculation, we obtain the final photon distribution in thermal equilibrium with the atom

$$P(0) = |a_0^+|^2 \cos^2 \theta_0 + |a_0^-|^2 \sin^2 \theta_0, \quad (14)$$

$$P(n) = |a_n^+|^2 \cos^2 \theta_n + |a_n^-|^2 \sin^2 \theta_n + |a_{n-1}^+|^2 \sin^2 \theta_{n-1} + |a_{n-1}^-|^2 \cos^2 \theta_{n-1}, \quad (15)$$

for  $n \geq 1$ . This final photon distribution depends both of the initial state of the atom as well as of the initial photon distribution,  $\{|C_n|^2\}$ . We can easily calculate the probability of finding the atom in its excited (ground) state, independently of the number of photons in the system, *i.e.*

$$P_e \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sum_{n=0}^{\infty} |\langle n, e | \Psi(t) \rangle|^2 dt = \sum_{n=0}^{\infty} |a_n^+|^2 \cos^2 \theta_n + |a_n^-|^2 \sin^2 \theta_n, \quad (16)$$

$$P_f = 1 - P_e. \quad (17)$$

Note that, using Eq.(12),

$$|a_n^+|^2 = \mathcal{N}^2 \left( C_n^2 \cos^2 \theta_n \cos^2 \frac{\gamma}{2} + \frac{1}{2} C_n C_{n+1} \sin 2\theta_n \sin \gamma \cos \varphi + C_{n+1}^2 \sin^2 \theta_n \sin^2 \frac{\gamma}{2} \right), \quad (18)$$

$$|a_n^-|^2 = \mathcal{N}^2 \left( C_n^2 \sin^2 \theta_n \cos^2 \frac{\gamma}{2} - \frac{1}{2} C_n C_{n+1} \sin 2\theta_n \sin \gamma \cos \varphi + C_{n+1}^2 \cos^2 \theta_n \sin^2 \frac{\gamma}{2} \right), \quad (19)$$

then, the above probabilities are dependent both of the initial distribution of photons and of the initial state of the atom .

In the asymptotic equilibrium, the average number of photons,  $\langle n \rangle = \sum_{m=-1}^{\infty} nP(n)$ , also can be calculated. Using Eqs.(14), (15), (18), (19), it is straightforward to show that and the average number of photons in the equilibrium satisfies

$$\langle n \rangle = P_f + \left( \bar{n} - (1 - e^{-\bar{n}}) \sin^2 \frac{\gamma}{2} \right) / \left( 1 - e^{-\bar{n}} \sin^2 \frac{\gamma}{2} \right). \quad (20)$$

In the case  $\bar{n} \gg 1$ , it is easy to show that the difference  $\Delta n \equiv \langle n \rangle - \bar{n}$  is

$$\Delta n \cong P_f - \sin^2 \frac{\gamma}{2}, \quad (21)$$

so that, as it could be expected, the average number of photons does not change appreciably because of the presence of the atom. The probability of finding the atom in its excited (ground) state in the resonant case is

$$P_e = P_f = \frac{1}{2}, \quad (22)$$

independent of the atomic initial conditions. In this case  $\delta = 0$  and  $\theta_n = \pi/4$  for all  $n$ , see Eqs.(8), (16) and (17).

In the case  $\bar{n} \sim 0$  we have the vacuum of the optical cavity and Eq.(20) is reduced to

$$\langle n \rangle \cong P_f. \quad (23)$$

In this case, we can study the thermal effects in the JCM using the already mentioned TFD formalism, according to which there is a correspondence between the ensemble averages of statistical mechanics and the vacuum expectation values of quantum field theory [14]. This formalism was applied to the JCM in Refs. [12,13]. Ref.[13] obtains an explicit expression for the atomic state population as a function of temperature, namely

$$P_e = \frac{e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} + e^{-\beta\hbar\omega/2}}, \quad (24)$$

$$P_f = \frac{e^{-\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} + e^{-\beta\hbar\omega/2}}. \quad (25)$$

where  $\beta = (k_B T)^{-1}$ , with  $k_B$  the Boltzmann constant.

### 3 Entanglement and temperature

The unitary evolution of the JCM generates entanglement between the number of photons and the state of the atom. To characterize this entanglement we

start with the von Neumann entropy which is the quantum analogue of the Gibbs entropy

$$S_N(\rho) = -\text{tr}(\rho \log \rho). \quad (26)$$

where  $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$  is the density matrix of the quantum system. Due to the unitary dynamics of the JCM the system remains in a pure state and this entropy vanishes. However, for these pure states the entanglement between the state of the atom and the number of photons can be quantified by the associated von Neumann entropy for the reduced density operator that defines the entropy of entanglement

$$S(\rho) = -\text{tr}(\rho_c \log \rho_c), \quad (27)$$

where here we define  $\rho_c$  as

$$\rho_c \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sum_{n=0}^{\infty} \langle n | \rho(t) | n \rangle, \quad (28)$$

and the partial trace is taken over the atom states. Using the wave function Eq. (9), together with Eq. (28), we obtain the reduced density operator

$$\rho_c = \begin{pmatrix} P_e & 0 \\ 0 & P_f \end{pmatrix}, \quad (29)$$

Equation (29) is diagonal and we define the eigenvalues

$$\Lambda_+ \equiv P_e, \quad (30)$$

$$\Lambda_- \equiv P_f, \quad (31)$$

For a given value of  $\delta \neq 0$ , these eigenvalues depend on the initial condition of the atom through the angles  $\gamma$  and  $\varphi$  and also on the initial photon distribution, see Eqs.(16) and (17).

In the resonant case,  $\delta = 0$ , the system stays in the degenerate case  $\Lambda_+ = \Lambda_- = 1/2$ , and in this case the entanglement entropy is maximum,  $S(\rho) = 1$ . If we want a more complete description of this equilibrium in the asymptotic limit, we need to know whether it is possible to associate a temperature to this entanglement entropy. In order to answer this question, we follow references [1,2,3]. According to the procedure described in them, it is necessary to connect the eigenvalues of  $\rho_c$  with an unknown associated Hamiltonian operator  $H_c$ . To obtain this connection we use the quantum Brownian motion model of Ref.[18]. There one considers the system associated with the atomic degrees of freedom, (atomic states characterized by the density matrix  $\rho_c$ ) in thermal contact (entanglement) with the bath system associated to the photons. The number of photons can be thought as an external degree of freedom, in interaction

with the internal degree of freedom, the atom state. This picture is equivalent to the thermal contact between one system and its heat bath. In equilibrium

$$[H_c, \rho_c] = 0, \quad (32)$$

should be satisfied. As a consequence, in the asymptotic regime the density operator  $\rho_c$  is an explicit function of a time-independent Hamiltonian operator. If we note by  $\{|\Phi_{\pm}\rangle\}$  the set of eigenfunctions of the density matrix, the operators  $H_c$  and  $\rho_c$  are both diagonal in this basis. Therefore the eigenvalues  $\Lambda_{\pm}$  depend on the corresponding eigenvalues of  $H_c$ . We denote this set of eigenvalues by  $\{\epsilon_{\pm}\}$ ; they can be interpreted as the possible values of the entanglement energy. This interpretation agrees with the fact that  $\Lambda_{\pm}$  is the probability that the system is in the eigenstate  $|\Phi_{\pm}\rangle$ . The precise dependence between  $\Lambda_{\pm}$  and  $\epsilon_{\pm} = \pm\epsilon$  is determined by the type of ensemble. In our case, this equilibrium corresponds to a quantum canonical ensemble. Therefore we propose that

$$\Lambda_{\pm} \equiv \frac{e^{\pm\beta\epsilon}}{e^{\beta\epsilon} + e^{-\beta\epsilon}}, \quad (33)$$

which defines the entanglement temperature  $T \equiv 1/(k_B\beta)$ . Of course in Eq.(33) only the ratio  $\epsilon/T$  is well defined, however we choose to introduce this temperature as this concept strengthens the idea of asymptotic equilibrium between the photon distribution and the state of the atom. Note that while temperature makes sense only in the mentioned equilibrium state, the entropy concept can be introduced without such a restriction.

The probability that a state chosen at random from the ensemble  $\{\Phi_+, \Phi_-\}$ , possesses an energy  $\epsilon$  is determined by the Boltzmann factor  $e^{-\beta\epsilon}$ . Let us call  $\widetilde{\rho}_c$  the diagonal expression of the density operator  $\rho_c$ , in the case treated in this paper  $\widetilde{\rho}_c = \rho_c$ , then

$$\widetilde{\rho}_c = \begin{pmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{pmatrix} = \frac{1}{e^{\beta\epsilon} + e^{-\beta\epsilon}} \begin{pmatrix} e^{\beta\epsilon} & 0 \\ 0 & e^{-\beta\epsilon} \end{pmatrix}. \quad (34)$$

This operator is formally the same density operator that corresponds to an electron which possesses an intrinsic spin and a magnetic moment in an external magnetic field [17]. In general the Hilbert space of a quantum mechanical model factors as a tensor product  $\mathcal{H}_{sys} \otimes \mathcal{H}_{env}$  of the spaces describing the degrees of freedom of the system and the environment. The evolution of the system is determined by the reduced density operator that results from taking the trace over  $\mathcal{H}_{env}$  to obtain  $\varrho_{sys} = \text{tr}_{env}(\rho)$ . The simple models studied in Refs. [19,20,21] show how the correlations of a quantum system with other systems may cause one of its observable to behave in a classical manner. In this sense the fact that the partial trace over the photon number leads to a system effectively in thermal equilibrium, agrees with those previous results. Starting from Eq.(34) it is possible to build the thermodynamics for the JCM



entanglement, where the partition function of the system is then given by

$$\mathcal{Z} = e^{\beta\epsilon} + e^{-\beta\epsilon} = 2 \cosh(\beta\epsilon), \quad (35)$$

and the entanglement temperature is determined by

$$T = \frac{2\epsilon}{k_B \ln \left( \frac{\Lambda_+}{\Lambda_-} \right)}. \quad (36)$$

When the system is in the resonance,  $\Lambda_{\pm} = 1/2$ , it has maximum entropy and the entanglement temperature is also maximum,  $T = \infty$ . It is important to point out that from the partition function Eq.(35) it is possible to develop all the entanglement thermodynamics, in other words, build the Hemholtz free energy, the internal energy and the entropy. As it should be expected, the entropy obtained from the partition function agrees with the previous Shannon expression, Eq.(40). Using Eqs.(30), (31) and (33), it is straightforward to show that

$$P_e = \frac{e^{\beta\epsilon}}{e^{\beta\epsilon} + e^{-\beta\epsilon}}, \quad (37)$$

$$P_f = \frac{e^{-\beta\epsilon}}{e^{\beta\epsilon} + e^{-\beta\epsilon}}. \quad (38)$$

It is interesting to make the connection between Eqs. (37) and (38) with the equivalent expressions obtained within the TFD formalism, Eqs. (24) and (25). From a straightforward comparison of these equations, we can determine explicitly the eigenvalues of  $H_{\tilde{\rho}}$

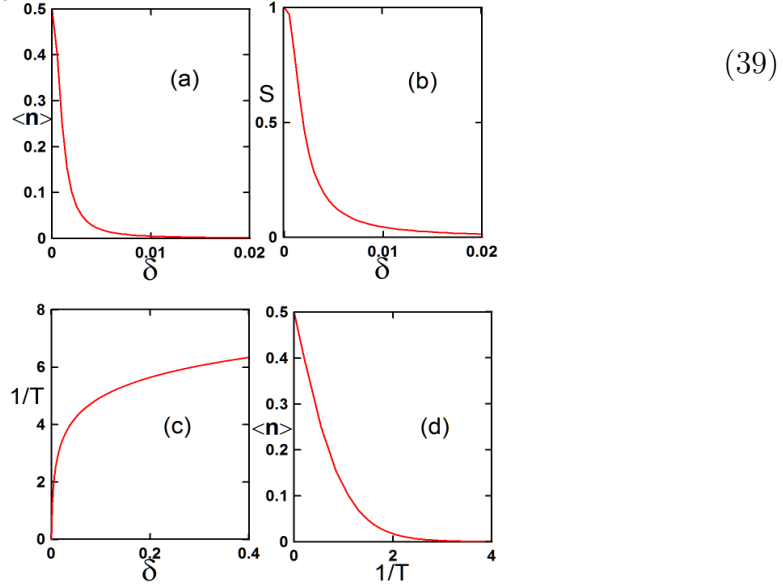


Fig. 1. (a) Change in the average number of photons, (b) dimensionless entropy, and (c) inverse of the dimensionless entanglement temperature,  $1/T$ , as a function of  $\delta$ . (d) Dependence of  $\langle n \rangle$  on  $1/T$ . In all these calculations we took  $g = 0.001$  and  $\bar{n} = 0$ .

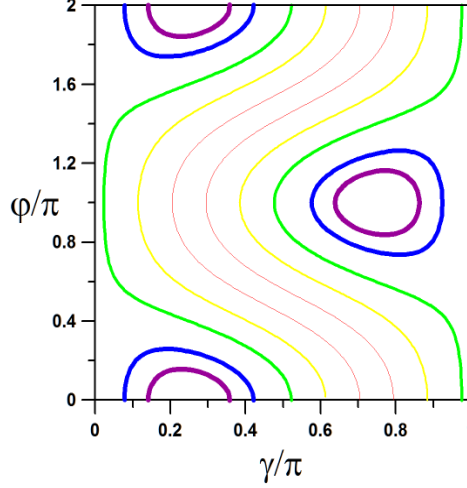


Fig. 2. (Color online) Five isothermal curves as function of the initial conditions, the dimensionless angles  $\gamma$  and  $\varphi$ . Each curve is characterized by its thickness (and color). The values of  $\beta \propto 1/T$ , from the thickest to the thinnest line are: 0.8 (purple), 0.7 (blue), 0.5 (green), 0.3 (yellow) and 0.1 (red). The isotherm corresponding to  $\beta = 0$  ( $T = \infty$ ) is situated between the two red isotherms. Here  $g = 0.001, \bar{n} = 100, \delta = 0.01$ .

and we conclude that the thermal vacuum temperature as defined in TFD and the entanglement temperature are essentially the same quantity, at least in the case of the JCM. The entanglement entropy, Eq.(27), can be expressed through the eigenvalues of  $\rho_c$  as

$$S(\rho) = -\Lambda_+ \log_2 \Lambda_+ - \Lambda_- \log_2 \Lambda_-. \quad (40)$$

Our results, for  $\bar{n} \sim 0$ , are presented graphically in Fig. 1, which shows the

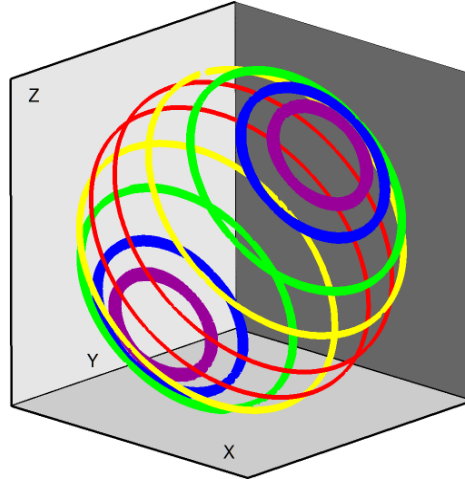


Fig. 3. (Color online) The isothermal curves of Fig. 2 are shown on the Bloch sphere. The rotation axis of the figure is in the direction  $(1, 0, 1)$ . The temperature of the entanglement between the atom and the photons is determined by the atomic initial condition.

behavior of  $\langle n \rangle$  as a function of the resonant parameter  $\delta$ . Figure 1(a) shows that for any value of  $\delta$ ,  $\langle n \rangle < 1/2$ . Figure 1(b) shows that the entanglement entropy has its maximum value for the resonant case *i.e.*  $S = 1$  when  $\delta = 0$ . The maximum value of the entropy (maximum disorder) is achieved when the dimensionless entanglement temperature is  $T = \infty$ , see Fig. 1(c). Under these conditions the atom excitation probability behaves as a classical Markov process. The relation between  $\langle n \rangle$  and the inverse of the dimensionless entanglement temperature, shown in Fig. 1(d), illustrates that, for large values of the entanglement temperature, a proportionality exists between  $\langle n \rangle$  and  $1/T$ . On the other hand for small values of  $T$ ,  $\langle n \rangle$  saturates to a constant value 0. These results are in accordance with Eq.(20). For  $\bar{n} \gg 1$ , figures 2 and 3 show the isotherms for the entanglement temperature as a function of the atomic initial condition, Eq.(10). In Fig. 2 the initial position is defined through the angles  $\gamma$  and  $\varphi$  and in Fig. 3 it is defined through the position on the Bloch sphere.

## 4 Conclusions

We studied the unitary evolution of the JCM in a composite Hilbert space. In particular, we considered the entanglement between a single two-level atom and a photon distribution characterized by a single frequency.

The JCM probability distribution does not converge with time. However, the asymptotic limit of the time-average of the probability distributions does converge. Thus the system establishes a stationary entanglement between the state of the atom and the photon distribution that allows to develop a thermodynamic theory. The asymptotic reduced density operator is used to introduce the entanglement thermodynamic functions in the canonical equilibrium. These thermodynamic functions characterize the asymptotic entanglement. The JCM can be put into correspondence with a quantum walk on the line, where the internal degree of freedom (the state of the atom) is entangled with the position degree of freedom (the photon number).

The difference between the average number of photons, with and without photon-atom interaction, depends on the entanglement temperature, and we show a connection between the entanglement temperature and the electromagnetic field intensity in the optical cavity.

We remark that in the case studied in this paper the entanglement temperature between the photons and the atom coincides with the temperature obtained within the TFD thermodynamics for the JCM. This implies that the entanglement temperature is amenable to direct measurements. We acknowl-

edge the support from PEDECIBA (Programa de Desarrollo de las Ciencias Básicas) and ANII (Agencia Nacional de Investigación e Innovación) (FCE-2-211-1-6281, Uruguay), and the CAPES-UdelaR collaboration program. A.R. acknowledges the stimulant comments of Victor Micenmacher.

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